Linear equations of motion for massless particles of any spin in any even dimensional spaces

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Abstract

It is proven that the Poincaré symmetry determines equations of motion, which are for massless particles of any spin in d-dimensional spaces linear in the momentum: $(W^a = \alpha p^a)|\Phi\rangle$ with W^a the generalized Pauli-Ljubanski vector. The proof is made only for even d and for fields with no gauge symmetry. We comment on a few examples.

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I. INTRODUCTION.

The ordinary space-time which we seem to experience is four-dimensional. However, besides the ordinary space-time, the internal space of spin and charges is equally important. Without spins and charges the World would manifest no dynamics - noninteracting chargeless scalar fields could form no matter. Theories of strings and membranes [1] as well as Kaluza-Klein theories [2] are predicting more than four-dimensional ordinary space-time. One of us [3–10] has proposed the approach, which describes geometrically not only the ordinary space-time but also the internal space of spins and charges, unifying spins and charges. In this approach the internal space is described by a vector space spanned over the anticommuting coordinate space of the same dimension d as that of ordinary space-time. To describe the physics of the Standard model both spaces, the one of commuting and the one of anticommuting coordinates, have to be more than four-dimensional. In d-dimensional space-time only spin degrees of freedom exists. It is the appropriate break of symmetry, which in four-dimensional subspace makes the spin manifesting itself as the spin and the known charges. In all theories with $d > 4$ the question arises why Nature has made a choice of four-dimensional subspace with one time and three space coordinates and with the particular choice of charges beside the spin degree of freedom for either spinors or vectors.

Mankoč and Nielsen [11,12] proved that in d-dimensional spaces, with even d, the spin degrees of freedom require q time and $(d-q)$ space dimensions, with q which has to be even. Accordingly in four-dimensional space Nature could only make a choice of the Minkowski metric. This proof was made under the assumption that equations of motion are for massless fields of any spin linear in the d-momentum $p^a,\ a = 0, 1, 2, 3, 5, \ldots, d$. (In addition, also the Hermiticity of the equations of motion operator as well as that this operator operates within an irreducible representation of the Lorentz group was required.) Our experiences tell us that equations of motion of all known massless fields are linear in the four-momentum $p^a, a = 0, 1, 2, 3$. We are refering [13] to the Dirac equation of motion for massless spinor fields and the Maxwell or Maxwell-like equations of motion for massless vectorial fields. One

of us together with A. Borštnik [8] has shown that the Weyl-like equations exist not only for spinors but also for vectors.

In this paper we present the proof that equations of motion for free massless particles in even-dimensional spaces, if manifesting the Poincaré symmetry, are linear in the p^a-momentum. For four dimensional space-time Wigner [14] clarified that as far as masses and spins are concerned (point-like) particles can be described by their properties under transformations of the Poincaré group. The classification of particles with respect to the unitary discrete representations of the Poincaré group can be found in Weinberg [15], for example. In his language equations of motion are those equations that constrain a solution space to a certain Poincaré group representation. For spinors this leads to the Dirac equation and for vectors to the Maxwell equations [16]. The aim of this paper is to use similar techniques for massless particles in general even-dimensional spaces. We prove that in even-dimensional spaces, for any $d = 2n$, free massless fields $|\Phi\rangle$

$$(p^a p_a = 0)\ |\Phi\rangle, \quad a = 0, 1, 2, 3, 5, \cdots, 2n \tag{1}$$

of any spin satisfy equations of motion which are linear in a d-momentum $p^a = (p^0, \overrightarrow{p})$

$$(W^a = \alpha p^a)\ |\Phi\rangle, \quad a = 0, 1, 2, 3, 5, \ldots, d, \tag{2}$$

with $\alpha = \frac{\overrightarrow{S}\cdot\overrightarrow{p}}{|p^0|}$ to be determined in this paper[1]. For spinors, which will be determined in Eq.(7), $\alpha = \pm\frac{1}{2}$. The proof is made only for fields with no gauge symmetry and with a nonzero value of the *handedness* operator [4] $\Gamma^{(int)}$

[1]After this paper appeared on hep-th W. Siegel let us know that equations $(S^{ab}p_b + wp^a = 0)|\Phi\rangle$ are the linear equations in p^a-momentum, as well as in S^{ab}, which all irreducible representations of massless fields in any d obey, and that these equations can be found in his book [20] and in his paper [19]. Following derivations of this paper in section (IV) one easily proves that solutions of Siegel's equations belong to irreducible representations of the Poincaré group. The proof is simpler than for our equations $(W^a = \alpha p^a)|\Phi\rangle$. Both equations are of course equivalent. Following our derivations one finds that the constant w in the Siegel's equation is $w = l_n$, with l_n defined

$$\Gamma^{(int)} = \beta\, \varepsilon_{a_1 a_2 \dots a_{d-1} a_d} S^{a_1 a_2} \dots S^{a_{d-1} a_d}, \tag{3}$$

which commutes with all the generators of the Poincaré group. We choose β so that $\Gamma^{(int)} = \pm 1$. For spinors (Eq.(7)) $\beta = i \frac{2^n}{(d!)}$, while for fields of a general spin β will be determined in section III. Operators S^{ab} are the generators of the Lorentz group $SO(1, d-1)$ in internal space, which is the space of spin degrees of freedom. In Eq.(2) W^a is the generalized Pauli-Ljubanski [4] d-vector

$$W^a = \rho\, \varepsilon^{ab}{}_{a_1 a_2 \dots a_{d-3} a_{d-2}} p_b S^{a_1 a_2} \dots S^{a_{d-3} a_{d-2}}. \tag{4}$$

We define [12] S^i as the $d-1$ vector

$$S^i = \rho\, \varepsilon^{0i}{}_{a_1 a_2 \cdots a_{d-3} a_{d-2}} S^{a_1 a_2} \cdots S^{a_{d-3} a_{d-2}} \tag{5}$$

and determine the value of ρ so that the eigenvalues of S^i are independent of a dimension and the same as in four-dimensional space. We will show that this dictates the choice $\rho = \frac{2^{n-2}}{(d-2)!}$ for spinors ($S^i = \pm \frac{1}{2}$) and $\rho = \frac{1}{2^{n-1}(n-1)!^2}$ for vectors ($S^i = \pm 1, 0$). Eq.(2) also guarantees that Eq.(1) is fulfilled for all massless fields with no gauge freedom and with nonzero spin.

We prove that for spinors in d-dimensional space Eq.(2) is equivalent to the equation

$$(\Gamma^{(int)} p^0 = \frac{1}{|\alpha|}\ \overrightarrow{S} \cdot \overrightarrow{p})|\Phi\rangle. \tag{6}$$

with $\frac{1}{|\alpha|}$ equal to 2 for any d=2n, while for a general spin Eq.(2) may impose additional conditions on the field.

We recognize the generators S^{ab} to be of the spinorial character, if they fulfil the relation

$$\{S^{ab}, S^{ac}\} = \frac{1}{2}\, \eta^{aa}\, \eta^{bc}, \ \text{no summation over a}, \tag{7}$$

with $\{A, B\} = AB + BA$.

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in Eq.(22). One derives our equation from the Siegel's one for even d after some rather tedious calculations if multiplying it by $\varepsilon_{aca_1 a_2 \dots a_{d-3} a_{d-2}} S^{a_1 a_2} \cdots S^{a_{d-3} a_{d-2}}$. The way how we present the linear equations of motion has several obvious advantages, which reader will easily find by himself.

In this paper the metric is, independently of the dimension, assumed for simplicity to be the Minkowski metric with $\eta^{ab} = \delta^{ab}(-1)^A,\; A = 0,$ for $a = 0$ and $A = 1$, otherwise.

The paper is arranged as follows:

We first present in section II the infinitesimal generators of the Poincaré group in d-dimensional spaces, the corresponding algebra and the Casimirs. We define appropriate d-vectors and $(d-1)$-vectors (some of them can be defined only in even-dimensional spaces), and present their properties and their commutation (and for spinors anticommutation) relations.

In section III we review representations of the Lorentz group and determine the parameter β of Eq.(3), specifying the Casimir $\Gamma^{(int)}$ for a general spin.

In section IV we define the generators of the little group and the constraints, which the generators of the little group have to fulfil in order to define discrete representations of the Poincaré group. We determine the factor α in Eq.(2), so that the equation Eq.(2) holds on discrete representations of the Poincaré group and represents accordingly a promising candidate for equations of motion.

In section V we investigate the solutions of Eq.(2) on spaces with $\Gamma^{(int)} \neq 0$. We show that they form irreducible representations of the Poincaré group (barring the sign of energy degeneracy) and in this sense we prove that Eq.(2) is the equation of motion.

At the end we comment in section VI on spinorial representations and on vectorial (we shall define these representations later) representations in any $d = 2n$-dimensional spaces.

II. POINCARÉ SYMMETRY.

The generators of the Poincaré group, that is the generators of translations p^a and the generators of the Lorentz transformations M^{ab} (which form the Lorentz group), fulfil in any dimension d, even or odd, the commutation relations:

$$
\begin{aligned}
[p^a, p^b] &= 0, \\
[M^{ab}, M^{cd}] &= i(\eta^{ad}\, M^{bc} + \eta^{bc}\, M^{ad} - \eta^{ac}\, M^{bd} - \eta^{bd}\, M^{ac}),
\end{aligned}
$$

$$[M^{ab}, p^c] = i(\eta^{bc}\, p^a - \eta^{ac}\, p^b). \tag{8}$$

The generators of the Lorentz transformations of the internal group S^{ab} ($M^{ab} = L^{ab} + S^{ab}$, with $L^{ab} = x^a p^b - x^b p^a$) fulfil the same commutation relations as M^{ab} (as do also L^{ab} by themselves) and commute with p^a (commutation relations for $[L^{ab}, p^c]$ are the same as for M^{ab}).

There are n commuting operators of the Lorentz group in d-dimensional spaces, for either $d = 2n+1$ or $d = 2n$, namely $M^{01}, M^{23}, \cdots, M^{d-1\, d}$ (or any other set of generators with all different indices) and accordingly n Casimirs of the Lorentz group. There are two Casimirs of the Lorentz group [4,6], which are easily found to be:

$$\begin{aligned} M^2 :&= \frac{1}{2}\, M^{ab} M_{ab} \quad \text{and} \\ \Gamma :&= \beta\, \varepsilon_{a_1 a_1 \cdots a_{2n-1} a_{2n}}\, M^{a_1 a_2} \cdots M^{a_{2n-1} a_{2n}}. \end{aligned} \tag{9}$$

The second Casimir of Eq.(9) can only be defined for $d = 2n$. We see that $\Gamma^{(int)}$ from Eq.(3) follows from Γ by replacing M^{ab} by S^{ab} and that $\Gamma^{(int)}$ commutes with all the generators of the Poincaré group.

With the help of Γ the generalized Pauli-Ljubanski vector as presented in Eq.(4) can be defined as

$$W^a = -i \frac{\rho}{2n\beta} [\Gamma, p^a], \tag{10}$$

with β and ρ from Eqs.(3, 4) for any even d. W^a in Eq.(10) can be as well expressed by S^{ab} instead by M^{ab}. One immediately finds that

$$W^a p_a = 0. \tag{11}$$

Lemma II.1: $W^a W_a$ is the Casimir of the Poincaré group and so is $p^a p_a$.

Proof: The proof goes, by taking into account Eq.(10) and the Lie algebra of the Poincaré group, as follows

$$[W^a W_a, M^{bc}] = -i\frac{\rho}{2n\beta}\{W_a\,[[\Gamma, p^a], M^{bc}] + [[\Gamma, p^a], M^{bc}]\,W_a\} =$$
$$= -i\frac{\rho}{2n\beta}\{W_a\,[\Gamma, [p^a, M^{bc}]] + [\Gamma, [p^a, M^{bc}]]\,W_a\} = 0. \tag{12}$$

The proof that $W^a W_a$ commutes with p^b is obvious due to Eq. (4), while the proof that $p^a p_a$ commutes with M^{bc} is straightforward. ■

The following commutation relations follow

$$[p^a, W^b] = 0,$$
$$[M^{ab}, W^c] = -i(\eta^{ac} W^b - \eta^{bc} W^a),$$
$$[\frac{1}{p^0}, W^a] = \frac{i}{(p^0)^2}\,(\eta^{a0} p^b - \eta^{b0} p^a). \tag{13}$$

To prove the first equation of Eqs.(13) is straightforward. To prove the second one, Eq.(10) as well as Jacobi identity $[M^{ab}, [\Gamma, p^c]] + [p^c, [M^{ab}, \Gamma]] + [\Gamma, [p^c, M^{ab}]] = 0$ have to be taken into account. To prove the third of the above equations the equations $[M^{ab}, \frac{p^0}{p^0}] = 0 = \frac{1}{p^0}\,[M^{ab}, p^0] + [M^{ab}, \frac{1}{p^0}]p^0$ have to be taken into account. This type of a proof was used also in the ref.([16]).

We present, for the spinorial case only, the commutation and anticommutation relations for $(d-1)$-vectors S^i, which are defined in Eq.(5)

$$[S^i, S^j] = i(-1)^{i-j} S^{ij}, \quad \{S^i, S^j\} = \frac{1}{4}\eta^{ij}. \tag{14}$$

III. REPRESENTATIONS OF LORENTZ GROUP IN INTERNAL SPACE.

In this section we introduce the notation for ireducible representations of the Lorentz group $SO(1, d-1)$ (or $SO(d)$) for $d = 2n$, denoting an irreducible representation by the weight of the dominant weight state of the representation. This exposition of the subject follows those of [17], [18]. We also express the two Casimirs of Eq.(9) in terms of the dominant weight. We pay attention only to the internal degrees of freedom for the Lorentz group, that is to a spin.

The Lie algebra of $SO(1,d-1)$ is spanned by the generators S^{ab}, which satisfy the commutation relations of the second equation of Eqs.(8)

$$[S^{ab}, S^{cd}] = i(\eta^{ad}\, S^{bc} + \eta^{bc}\, S^{ad} - \eta^{ac}\, S^{bd} - \eta^{bd}\, S^{ac}). \tag{15}$$

We choose the n commuting operators of the Lorentz group $SO(1,d-1)$ as follows

$$-iS^{0d}, S^{12}, S^{35}, \ldots, S^{d-2\, d-1} \tag{16}$$

and call them $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n-1}$, respectively.

(Everything what follows will be valid also for the group $SO(d)$, provided that the generators $-iS^{01}, -iS^{02}, \ldots$ are corespondingly replaced, that is by $S^{01}, S^{02}, \ldots$, respectively.)

We say that a state $|\Phi_w\rangle$ has the weight $(w_0, w_1, w_2, \ldots, w_{n-1})$ if the following equations hold

$$\mathcal{C}_j |\Phi_w\rangle = w_j |\Phi_w\rangle, \quad j = 0, 1, \ldots, n-1. \tag{17}$$

According to the definition in Eq.(16), weight components $w_0, w_1, w_2, \ldots, w_{n-1}$ are always real numbers.

We introduce the raising and lowering operators

$$E_{jk}(\lambda, \mu) := \frac{1}{2}((-i)^{\delta_{j0}} S^{j_- k_-} + i\lambda S^{j_+ k_-} - i^{1+\delta_{j0}} \mu S^{j_- k_+} - \lambda\mu S^{j_+ k_+}), \tag{18}$$

with $0 \le j < k \le n-1,\ \lambda, \mu = \pm 1$ and $0_- = 0, 0_+ = d, 1_- = 1, 1_+ = 2, 2_- = 3, 2_+ = 5$ and so on. For these operators the following commutation relations hold

$$[E_{jk}(\lambda, \mu), \mathcal{C}_l] = (\delta_{jl}\lambda + \delta_{kl}\mu) E_{jk}(\lambda, \mu). \tag{19}$$

Therefore, if the state $|\Phi_w\rangle$ has the weight $(w_0, w_1, \ldots, w_{n-1})$ then the state $E_{jk}(\lambda, \mu)|\Phi_w\rangle$ has the weight $(\ldots, w_j + \lambda, \ldots, w_k + \mu, \ldots)$.

We now proceed to the definition of the dominant weight. We fix $q \in \{+1, -1\}$ and call the state $|\Phi_l\rangle$ with the property

$$E_{jk}(q, \pm 1)|\Phi_l\rangle = 0, \quad 0 \le j < k \le n-1 \tag{20}$$

the dominant weight state. This state is (up to a scalar multiple) uniquely determined by the irreducible representation in question and vice versa, knowing the dominant weight state of a particular representation (or any state of a particular representation), all the others are obtained by the application of the generators S^{ab}. We will denote the irreducible representation of the Lorentz group $SO(1,d-1)$ by the weight of the dominant weight state: $(l_n, l_{n-1}, \ldots, l_2, l_1)_q$ with the index q attached to distinguish which definition of the dominant weight state we are using. Numbers $l_n, l_{n-1}, \ldots, l_2, l_1$ are either all integer or all half integer and satisfy

$$l_n \geq l_{n-1} \geq \ldots \geq l_2 \geq |l_1| \tag{21}$$

in case $q = +1$ or

$$l_n \leq l_{n-1} \leq \ldots \leq l_2 \leq -|l_1| \tag{22}$$

in case $q = -1$.

The correspondence between the two notations is given by the fact that the following representations are equivalent

$$(l_n, l_{n-1}, \ldots, l_2, l_1)_{+1} = (-l_n, -l_{n-1}, \ldots, -l_2, (-1)^{n-1} l_1)_{-1}. \tag{23}$$

In section V we will find it useful to work with the notation $q = +1$ when dealing with the positive energy ($p^0 > 0$) representations and with $q = -1$ when dealing with the negative energy ($p^0 < 0$) representations.

For future use we reformulate the condition (20) which determines the dominant weight state. It follows from Eq.(18) that

$$\begin{aligned} - i(E_{0k}(q,+1) + E_{0k}(q,-1)) &= -S^{0k_-} + qS^{dk_-}, \\ -E_{0k}(q,+1) + E_{0k}(q,-1) &= -S^{0k_+} + qS^{dk_+} \end{aligned} \tag{24}$$

for $k = 1, 2, \ldots, n-1$. Eq.(20) then implies

$$(S^{0i} + qS^{id})|\Phi_l\rangle = 0, \quad i = 1, 2, 3, 5, \ldots, d-1. \tag{25}$$

Similarly, we obtain the following conditions

$$(S^{1i} + qiS^{2i})|\Phi_l\rangle = 0, \quad i = 3, 5, \ldots, d-1, \tag{26}$$

$$(S^{3i} + qiS^{5i})|\Phi_l\rangle = 0, \quad i = 6, 7 \ldots, d-1, \text{ and so on.} \tag{27}$$

By reversing this process we can also conclude that Eqs.(25), (26),(27) imply (20). They are therefore equivalent to condition (20).

We now determine the values of the two Casimirs of Eq.(9) for a particular irreducible representation $(l_n, l_{n-1}, \ldots, l_2, l_1)_q$. Since the Casimirs (Eq.(9)) are scalars it suffices to determine their value on the dominant weight state $|\Phi_l\rangle$. Since the dominant weight state satisfies Eqs.(25),(26),(27) which are not invariant to the permutations of $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$, we don't expect the values of Casimirs to be symmetric in $l_n, l_{n-1}, \ldots, l_1$. It will turn out this is indeed the case.

We now find for the Casimir M^2 (which we shall denote when useful by M_d^2) from Eq.(9)

$$M_d^2 = \frac{1}{2} S^{ab} S_{ab} = -(S^{0d})^2 - S^{0i}S^{0i} + S^{di}S^{di} + \frac{1}{2} S^{ij} S_{ij}. \tag{28}$$

We use Eq.(25) and the relation $S^{0i}S^{0i} - S^{di}S^{di} = (S^{0i} - qS^{di})(S^{0i} + qS^{di}) - q[S^{0i}, S^{di}]$ to find

$$\begin{aligned}
(S^{0i}S^{0i} - S^{di}S^{di})|\Phi_l\rangle &= iq(d-2)S^{0d}|\Phi_l\rangle, \quad \Longrightarrow \\
M_d^2|\Phi_l\rangle &= (-q(d-2)iS^{0d} - (S^{0d})^2 + \frac{1}{2}S^{ij}S_{ij})|\Phi_l\rangle, \\
M_d^2|\Phi_l\rangle &= ((d-2)ql_n + l_n^2 + M_{d-2}^2)|\Phi_l\rangle.
\end{aligned} \tag{29}$$

The operators S^{ij} belong to the Lie algebra of the subgroup $SO(d-2)$ of the group $SO(1, d-1)$ acting only on coordinates $1, 2, \ldots, d-1$. We may use Eq.(29) inductively to express the Casimir M_d^2

$$M_d^2|\Phi_l\rangle = (l_n(l_n + q(d-2)) + l_{n-1}(l_{n-1} + q(d-4)) + \ldots + l_2(l_2 + 2q) + l_1^2)|\Phi_l\rangle. \tag{30}$$

We proceed with the Casimir $\Gamma^{(int)}$ of Eq.(3) in a similar way as with M^2 (again sometimes using the subscript d to point out the dimension of space-time). We define

$$\tilde{\Gamma}_d^{(int)} := \Gamma^{(int)}/\beta = \varepsilon_{a_1 a_2 \ldots a_{d-1} a_d} S^{a_1 a_2} S^{a_3 a_4} \ldots \tag{31}$$

and note

$$\tilde{\Gamma}_d^{(int)}/(2n) = S^{0d} \varepsilon_{0 d a_1 a_2 \ldots a_{d-3} a_{d-2}} S^{a_1 a_2} S^{a_3 a_4} \ldots + $$
$$+2(n-1) \varepsilon_{0 i d j a_1 a_2 \ldots a_{d-5} a_{d-4}} S^{0i} S^{dj} S^{a_1 a_2} S^{a_3 a_4} \ldots . \tag{32}$$

Applying $\tilde{\Gamma}_d^{(int)}/(2n)$ on the dominant weight state and taking into account Eq.(25) we find

$$\begin{aligned}
(\tilde{\Gamma}_d^{(int)}/(2n) - S^{0d}(\varepsilon_{0 d a_1 a_2 \ldots a_{d-3} a_{d-2}} S^{a_1 a_2} S^{a_3 a_4} \ldots)|\Phi_l\rangle &= \\
= (n-1)(\varepsilon_{0 i d j a_1 a_2 \ldots a_{d-5} a_{d-4}} (S^{0i} S^{dj} - S^{0j} S^{di}) S^{a_1 a_2} S^{a_3 a_4} \ldots)|\Phi_l\rangle &= \\
= (n-1) q (\varepsilon_{0 i d j a_1 a_2 \ldots a_{d-5} a_{d-4}} (S^{0i} S^{0j} - S^{0j} S^{0i}) S^{a_1 a_2} S^{a_3 a_4} \ldots)|\Phi_l\rangle &= \\
= (n-1) q (\varepsilon_{0 i d j a_1 a_2 \ldots a_{d-5} a_{d-4}} [S^{0i}, S^{0j}] S^{a_1 a_2} S^{a_3 a_4} \ldots)|\Phi_l\rangle &= \\
= i(n-1) q (\varepsilon_{0 d i j a_1 a_2 \ldots a_{d-5} a_{d-4}} S^{ij} S^{a_1 a_2} S^{a_3 a_4} \ldots)|\Phi_l\rangle &.
\end{aligned}$$

It therefore follows that

$$\begin{aligned}
\tilde{\Gamma}_d^{(int)} |\Phi_l\rangle = 2ni(l_n + q(n-1))(\varepsilon_{0 d a_1 a_2 \ldots a_{d-5} a_{d-4}} S^{a_1 a_2} S^{a_3 a_4} \ldots)|\Phi_l\rangle &= \\
2ni(l_n + q(n-1)) \, \tilde{\Gamma}_{d-2}^{(int)} |\Phi_l\rangle &,
\end{aligned} \tag{33}$$

where we have taken into account that $\Gamma_{d-2}^{(int)}$ refers to the handedness operator for subgroup $SO(d-2) \leq SO(1, d-1)$ acting on coordinates $1, 2, \ldots, d-1$. We could repeat this process for $\Gamma_{d-2}^{(int)}$ with equations

$$(S^{1i} + iqS^{2i})|\Phi_l\rangle = 0, \quad i = 3, 5, \ldots, d-1 \tag{34}$$

taking place of Eqs.(25). We obtain

$$\begin{aligned}
\tilde{\Gamma}_{d-2}^{(int)} |\Phi_l\rangle = 2(n-1)(l_{n-1} + q(n-2))(\varepsilon_{0 d 1 2 a_1 a_2 \ldots a_{d-3} a_{d-2}} S^{a_1 a_2} S^{a_3 a_4} \ldots)|\Phi_l\rangle &= \\
= 2(n-1)(l_{n-1} + q(n-2)) \, \tilde{\Gamma}_{d-4}^{(int)} |\Phi_l\rangle &.
\end{aligned}$$

We note the absence of a factor i in the last equation. Repeating this process inductively we find

$$(\tilde{\Gamma}_d^{(int)} = 2^n n! i(l_n + q(n-1))(l_{n-1} + q(n-2)) \ldots (l_2 + q)l_1)|\Phi_l\rangle. \tag{35}$$

(Had we been dealing with the group $SO(d)$ the result would have been similar, the only difference being the absence of i in the previous equation.) We have therefore obtained

$$\Gamma^{(int)} \, |\Phi_l\rangle = 2^n n! i\beta \prod_{j=1}^{n} \, (l_j + q(j-1))|\Phi_l\rangle. \tag{36}$$

It follows from (35) that on spaces $(l_n, \ldots, l_1)_{+1}$ with nonzero handedness ($\tilde{\Gamma}_d^{(int)} \neq 0 \Leftrightarrow l_1 \neq 0$) we must take

$$\beta = \frac{i}{2^n n! (l_n + n - 1) \ldots (l_2 + 1)|l_1|}, \quad d = 2n, \tag{37}$$

to obtain $\Gamma^{(int)} = \pm 1$. In what follows we assume this choice of β has been made.

IV. LITTLE GROUP.

In this section we characterize the unitary massless discrete representations of the Poincaré group and work out some of their properties. In doing so we follow the little group method [15].

For momenta p^a appearing in an irreducible massless representations of the Poincaré group it holds

$$p^a p_a = 0, \quad p^0 > 0 \text{ or } p^0 < 0 \tag{38}$$

(we omit the trivial case $p^0 = 0$). We denote $r = \frac{p^0}{|p^0|}$. The Poincaré group representation is then characterized by the representation of the so-called *little group*, which is a subgroup of the Lorentz group leaving some fixed d-momentum $p^a = \mathbf{k}^a$ satisfying Eq.(38) unchanged.

Let us make the choice of $\mathbf{k}^a = (rk^0, 0, \cdots, 0, k^0)$, with $r = \pm 1, k^0 > 0$. The infinitesimal generators of the little group $\omega_{ab} M^{ab}$, ($\omega_{ij} = -\omega_{ji}$) can be found by requiring that, when operating on the d-vector $\mathbf{k}^a$, give zero so that accordingly the corresponding group transformations leave the d-vector $\mathbf{k}^a$ unchanged

$$e^{-i\frac{1}{2}\omega_{bc}M^{bc}}\,\mathbf{k}^a = \mathbf{k}^a, \quad \text{or equivalently}$$

$$\omega_{bc}M^{bc}\mathbf{k}^a = 0. \tag{39}$$

We see that Eq.(39) requires

$$0 = \begin{pmatrix} 0 & -\omega_{01} & \dots & -\omega_{0d} \\ \omega_{10} & 0 & \dots & \omega_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{d0} & \omega_{d1} & \dots & 0 \end{pmatrix} \begin{pmatrix} rk^0 \\ 0 \\ \vdots \\ k^0 \end{pmatrix} = \begin{pmatrix} \omega_{0d} \\ r\omega_{10} + \omega_{1d} \\ \vdots \\ \omega_{d0} \end{pmatrix} k^0 \tag{40}$$

and it follows

$$\omega_{0d} = 0,$$

$$\omega_{i0} + r\omega_{id} = 0, \quad i = 1, 2, 3, 5, \cdots d-1. \tag{41}$$

All $\omega_{ab}M^{ab}$ with ω_{ab} subject to conditions (41) form the Lie algebra of the little group. We choose the following basis of the little group Lie algebra

$$\Pi_i = M^{0i} + rM^{id}, \quad i = 1, 2, 3, 5, \cdots, d-1$$

$$\text{and all } \ M^{ij}, \quad i, j = 1, 2, 3, 5, \cdots, d-1. \tag{42}$$

One finds

$$[\Pi_i, \Pi_j] = 0, \quad [\Pi_i, M^{jk}] = i(\eta^{ij}\Pi_k - \eta^{ik}\Pi_j). \tag{43}$$

We are interested only in discrete representations of the Poincaré group. This means that the states in the representation space can be labeled by the momentum and an additional label for internal degrees of freedom, which can only have *discrete* values.

Lemma IV.1: For a discrete representation of the Poincaré group operators Π_i give zero.

Proof: We may arrange the representation space of the little group to be eigenvectors of the commuting operators Π_i of the little group[1]

[1]We can diagonalize Π_i since we are interested in unitary representations.

$$\Pi_i|\Phi_a\rangle = b^i{}_a|\Phi_a\rangle. \tag{44}$$

Here a stands for a set of quantum numbers. We make the rotation $e^{i\theta M^{ij}}$ on a state $|\Phi_a\rangle$ depending on the parameter θ and the operator M^{ij}, with the particular choice of i and j. Taking into account Eq.(43) we find

$$\begin{aligned}
\Pi_i e^{i\theta M^{ij}}|\Phi_a\rangle &= (b^i{}_a cos\theta - b^j{}_a sin\theta)e^{i\theta M^{ij}}|\Phi_a\rangle,\\
\Pi_j e^{i\theta M^{ij}}|\Phi_a\rangle &= (b^i{}_a sin\theta + b^j{}_a cos\theta)e^{i\theta M^{ij}}|\Phi_a\rangle.
\end{aligned} \tag{45}$$

It is obvious that the states $e^{i\theta M^{ij}}|\Phi_a\rangle$ produce a continous set of eigenvalues for $\Pi_1,\ldots,\Pi_{d-1}$, since θ is a continous parameter, which contradicts the discreteness of the label a. The only exception occurs when $b^i{}_a cos\theta - b^j{}_a sin\theta = b^i{}_a$ and $b^i{}_a sin\theta + b^j{}_a cos\theta = b^j{}_a$ for all θ. This is only possible if $b^i{}_a = b^j{}_a = 0$ for all a and each i, j. Therefore

$$\Pi_i|\Phi_a\rangle = 0. \tag{46}$$

■

We see that on the representation space of the little group with the choice $\mathbf{k}^a = (rk^0, o, \cdots, 0, k^0)$, on which $(L^{0i} + rL^{id})|\Phi_a\rangle = (x^0\mathbf{k}^i - x^i\mathbf{k}^0 + rx^i\mathbf{k}^d - rx^d\mathbf{k}^i)|\Phi_a\rangle = 0$ the following holds

$$\Pi_i|\Phi_a\rangle = \Pi_i^{(int)}|\Phi_a\rangle = 0, \quad \Pi_i^{(int)} = (S^{0i} + rS^{id}). \tag{47}$$

It follows then that the only little group generators, which are not necessarily zero on the representation space, are $M^{ij}, i, j = 1, 2, \ldots, d-1$ and they form the Lie algebra of $SO(d-2)$. We conclude that *the irreducible discrete representations of the Poincaré group in $d(=2n)$ dimensions for massless particles are determined by the irreducible representations of the group $SO(d-2)$. We will therefore denote the former with the same symbol as the latter with an additional index r for the energy sign: $(l_{n-1}, l_{n-2}, \ldots, l_2, l_1; r)_q$.*

The group $SO(d-2)$ has $\frac{(d-2)(d-3)}{2}$ generators and according to the commutation relations of Eq.(8), $(\frac{d}{2}-1)$ commuting generators and also $(\frac{d}{2}-1)$ quantum numbers which determine a state $|\Phi_a\rangle$.

We now show the validity of the Eq.(2).

Lemma IV.2: On the representation space of an irreducible massless representation of the Poincaré group $(l_{n-1},\dots,l_1;r)_q$ the equation (2) holds with

$$\alpha = -\rho r 2^{n-1}(n-1)!(l_{n-1}+q(n-2))\dots(l_2+q)l_1. \tag{48}$$

Proof: First, we prove the lemma on the representation space of the little group for the choice $p^a = (rk^0,0,\dots,0,k^0), r=\pm 1, k^0>0$. We begin with the cases $a=1,2,\dots,d-1$ in Eq.(2)

$$\begin{aligned}
W^a/\rho|\Phi_a\rangle &= (\varepsilon^{aa_1}{}_{a_2a_3\dots a_{d-1}}p_b M^{a_1a_2}M^{a_3a_4}\dots)|\Phi_a\rangle = \\
&= k^0(r\varepsilon^{a0}{}_{a_1a_2\dots a_{d-3}a_{d-2}}M^{a_1a_2}M^{a_3a_4}\dots - \varepsilon^{ad}{}_{a_1a_2\dots a_{d-3}a_{d-2}}M^{a_1a_2}M^{a_3a_4}\dots)|\Phi_a\rangle = \\
&= 2k^0(n-1)(r\varepsilon^{a0}{}_{da_1a_2\dots a_{d-3}}M^{da_1}M^{a_2a_3}\dots - \varepsilon^{ad}{}_{0a_1a_2\dots a_{d-3}}M^{0a_1}M^{a_2a_3}\dots)|\Phi_a\rangle = \\
&= 2k^0(n-1)(\varepsilon^{0d}{}_{aa_1a_2\dots a_{d-3}}(rM^{da_1}-M^{0a_1})M^{a_2a_3}\dots)|\Phi_a\rangle = \\
&= -2k^0(n-1)\boldsymbol{\alpha}^{ai}\Pi_i|\Phi_a\rangle
\end{aligned} \tag{49}$$

where

$$\boldsymbol{\alpha}^{ai} = \varepsilon^{0dai}{}_{a_1a_2\cdots a_{d-4}}M^{a_1a_2}\cdots M^{a_{d-5}a_{d-4}}, \quad [\Pi^i,\boldsymbol{\alpha}^{ai}]=0. \tag{50}$$

Eq.(46) concludes the proof for $a=1,2,\dots,d-1$.

We now turn to the case $a=0$

$$\begin{aligned}
W^0/\rho|\Phi_a\rangle &= (\varepsilon^{0b}{}_{a_1a_2\dots a_{d-3}a_{d-2}}p_b M^{a_1a_2}M^{a_3a_4}\dots)|\Phi_a\rangle = \\
&= -k^0(\varepsilon^{0d}{}_{a_1a_2\dots a_{d-3}a_{d-2}}M^{a_1a_2}M^{a_3a_4}\dots)|\Phi_a\rangle = (-r)p^0\tilde{\Gamma}_{d-2}|\Phi_a\rangle.
\end{aligned}$$

Eq.(48) holds according to the value of the handedness operator obtained in section III and applied to the group $SO(d-2)$. The case $a=d$ goes similarly

$$\begin{aligned}
W^d/\rho|\Phi_a\rangle &= (\varepsilon^{db}{}_{a_1a_2\dots a_{d-3}a_{d-2}}p_b M^{a_1a_2}M^{a_3a_4}\dots)|\Phi_a\rangle = \\
&= rk^0(\varepsilon^{d0}{}_{a_1a_2\dots a_{d-3}a_{d-2}}M^{a_1a_2}M^{a_3a_4}\dots)|\Phi_a\rangle = (-r)p^d\tilde{\Gamma}_{d-2}|\Phi_a\rangle.
\end{aligned}$$

Again, we point out that in all these derivations M^{ab} can be replaced by S^{ab}.

To prove Eq.(2) on the whole representation space, one only has to note that Eq.(2) is in covariant form and must therefore hold generally. To put this explicitly: the linear hull of the states $|\Phi'_a\rangle = U(\Lambda)|\Phi_a\rangle$, where $|\Phi_a\rangle$ runs through the representation space of our little group and $\Lambda \in SO(1,d-1)$ (here $U(\Lambda)$ denotes the unitary transformation belonging to the element $\Lambda \in SO(1,d-1)$ in our representation), is dense in the whole representation space. It is therefore sufficient to prove (2) for $|\Phi'_a\rangle$. This follows from

$$(W^a - \alpha p^a)|\Phi'_a\rangle = U(\Lambda)U(\Lambda)^{-1}(W^a - \alpha p^a)U(\Lambda)|\Phi_a\rangle =$$

$$= U(\Lambda)(\Lambda^{-1})_{ab}(W^b - \alpha p^b)|\Phi_a\rangle = 0.$$

■

It is proved that Eq.(2) with the parameter α from Eq.(48) holds on the Poincaré group representation $(l_{n-1}, \ldots, l_1; r)_q$. It is therefore a candidate for an equation of motion. Before it can be admitted to that status we have to investigate its solutions. We do that in the next section.

V. EQUATIONS OF MOTION.

In this section we investigate the solutions of the equations

$$(W^a = \alpha p^a)|\Phi\rangle, \quad p_a p^a|\Phi\rangle = 0, \tag{51}$$

on the space with internal Lorentz group $SO(1,d-1)$ representation $(l_n, l_{n-1}, \ldots, l_2, l_1)_{+1}$. Since Eq.(51) is in a covariant form it is clear that the space of its solutions forms a representation of the Poincaré group. It also corresponds to massless particles as indicated by the second equation in Eq.(51). What remains to be investigated is the irreducibility and discreteness of solutions.

For irreducibility, we will allow only degeneracy in the sign of p^0 (the energy) and no degeneracy as far the internal degrees of freedom are concerned, since we don't want the

same equation to describe particles of different spins. As we will see, to provide both this and the discreteness, the condition

$$\alpha = -\rho 2^{n-1}(n-1)!(l_{n-1} + (n-2)) \dots (l_2 + 1) l_1 \tag{52}$$

must be fulfilled. For $d = 4$ this is also the sufficient condition, while for dimensions $d \geq 6$ for sufficiency the following condition must be added

$$l_1 \neq 0 \Longleftrightarrow \Gamma^{(int)} \neq 0. \tag{53}$$

As shown in section IV, due to the covariant form of Eq.(51) the investigation of the Poincaré representations formed by the solutions of Eq.(51) reduces to the investigation of the $SO(d-2)$-representations formed by the solutions of Eq.(51) with $p^a = \mathbf{k}^a = (rk^0, 0, \dots, 0, k^0), r = \pm 1, k^0 > 0$.

We begin with the following lemma.

Lemma V.1 : On the space with internal Lorentz group $SO(1, d-1)$ representation $(l_n, l_{n-1}, \dots, l_2, l_1)_r$, states satisfying the discreteness condition of Eq.(47) for the little group, form the $SO(d-2)$-irreducible representation space $(l_{n-1}, \dots, l_1)_r$, where the subgroup $SO(d-2) \leq SO(1, d-1)$ acts on coordinates $1, 2, \dots, d-1$.

Proof: Eqs.(43) imply that the space of solutions $\{|\Phi_a\rangle\}$ of Eq.(47) forms an $SO(d-2)$-invariant space, where $SO(d-2) \leq SO(1, d-1)$ acts on coordinates $1, 2, \dots, d-1$. To prove our lemma we must show that $\{|\Phi_a\rangle\}$ is irreducible and corresponds to the $SO(d-2)$-representation $(l_{n-1}, \dots, l_1)_r$. We do this by choosing any state $|\Phi_a\rangle \in \{|\Phi_a\rangle\}$ with a $SO(d-2)$-dominant weight (with $q = r$) and proving both that it is unique (up to a scalar multiple) and has $SO(d-2)$-weight $(l_{n-1}, \dots, l_1)$. This is possible because the $SO(1, d-1)$-dominant weight state is in $\{|\Phi_a\rangle\}$, as shown by Eqs.(25) with $q = r$, Eq.(46) and Eq.(47). The set of states $\{|\Phi_a\rangle\}$ is therefore not trivial. Since $|\Phi_a\rangle$ has a $SO(d-2)$-dominant weight, Eqs.(26) and (27) must hold with $q = r$. Eq.(47) then implies Eq.(25) for $q = r$. Therefore, the state $|\Phi_a\rangle$ has a $SO(1, d-1)$-dominant weight (with $q = r$) and is therefore unique (up to a scalar

multiple). Thus we have also proven that the corresponding representation of the Poincaré group is determined by all but the first dominant weight component of the internal Lorentz group representation. It is therefore $(l_{n-1}, l_{n-2}, \dots, l_2, l_1)_r$ as was to be shown. ■

Given any representation of the internal Lorentz group, say $(l_n, \dots, l_1)_{+1}$, the only possible discrete Poincaré representation are given by the previous lemma and Eq.(23): they are

$$(l_{n-1}, \dots, l_1; +1)_{+1} \quad \text{and} \quad (-l_{n-1}, \dots, -l_2, (-1)^{n-1} l_1; -1)_{-1} = (l_{n-1}, \dots, l_2, -l_1; -1)_{+1}. \tag{54}$$

We note that in four dimensions in our notation $(\sigma; r)_q$ of the Poincaré group representation the quantum number σ is helicity since it is the eigenvalue of the operator S^{12}. We have therefore generalized the known result in four dimensions which states $\sigma = r l_1$ ([15], [16]) to general even-dimensional spaces.

So, according to Eq.(48) if we are to hope for the solutions of Eq.(51) to form a discrete Poincaré representation we must have $\alpha = -\rho r 2^{n-1}(n-1)!(l_n + n - 1)\dots(l_2+1)(rl_1)$, which is exactly Eq.(52).

We now answer the question: when are the solutions of the equations

$$(W^a = \alpha p^a)|\Phi\rangle, \quad \alpha = -\rho 2^{n-1}(n-1)!(l_{n-1} + (n-2))\dots(l_2+1)l_1, \quad p_a p^a|\Phi\rangle = 0, \tag{55}$$

on the space with internal Lorentz group $SO(1, d-1)$ representation $(l_n, l_{n-1}, \dots, l_2, l_1)_{+1}$, exactly those described by Eq.(54) ? First, we deal with the simplest case of $d = 4$.

Lemma V.2 : On the space with internal Lorentz group $SO(1, 3)$ representation $(l_2, l_1)_{+1}$ the solutions of the Eq.(55) are exactly those of Eq.(54).

Proof: Again we choose $p^a = \mathbf{k}^a = (rk^0, 0, \dots, 0, k^0)$, $r = \pm 1$, $k^0 > 0$. In this case, the first equation of Eqs.(55), with $a = 1, 2$ reads as follows

$$(\Pi_2^{(int)} = 0)|\Phi\rangle, \tag{56}$$

$$(\Pi_1^{(int)} = 0)|\Phi\rangle. \tag{57}$$

These are exactly Eq.(47) and according to lemma V.1 their solutions are listed in Eq.(54). Finally, we note that $(W^0 = \alpha p^0)|\Phi\rangle$ and $(W^3 = \alpha p^3)|\Phi\rangle$ both give the equation $S^{12}|\Phi\rangle = rl_1|\Phi\rangle$ which agrees with Eq.(54) and therefore doesn't impose any additional constraints to the solutions of Eq.(54). This concludes the proof. ∎

We shall now prove the general case $d \geq 6$.

Lemma V.3 : On the space with the internal Lorentz group $SO(1, d-1)$ representation $(l_n, l_{n-1}, \ldots, l_1)_{+1}$, where

$$l_1 \neq 0 \Longleftrightarrow \Gamma^{(int)} \neq 0, \tag{58}$$

the solutions of Eq.(55) are exactly those of Eq.(54).

Proof: We prove the lemma for the standard choice of the d-momentum $p^a = (rk^0, 0, \ldots, 0, k^0)$, $r = \pm 1$, $k^0 > 0$. From the proof of lemma IV.2 we know that Eq.(55) reads as follows

$$\boldsymbol{\alpha}^{ij} \Pi_j^{(int)} |\Phi\rangle = 0, \tag{59}$$

$$-\rho \varepsilon^{0d}{}_{a_1 a_2 \ldots a_{d-3} a_{d-2}} S^{a_1 a_2} S^{a_3 a_4} \ldots |\Phi\rangle = \alpha |\Phi\rangle, \tag{60}$$

with $\boldsymbol{\alpha}^{ij} = \varepsilon^{0dij}{}_{a_1 a_2 \ldots a_{d-5} a_{d-4}} S^{a_1 a_2} S^{a_3 a_4} \ldots$ for $i, j = 1, 2, \ldots, d-1$. By lemma V.1 it is sufficient to show that every solution $|\Phi\rangle$ of Eqs.(59, 60) satisfies the condition

$$\Pi_i^{(int)} |\Phi\rangle = 0, \quad \text{for } i = 1, 2, \ldots, d-1. \tag{61}$$

In what follows we take the definition of dominant weights (Eq.(20)) with $q = r$. According to lemma V.1 and the discussion directly following this lemma the dominant weight of the group $SO(1, d-1)$ satisfies both Eqs.(59, 60), so that we conclude that the space $\mathcal{W}$ of solutions of Eq.(59) and Eq.(60) is nontrivial. It is also $SO(d-2)$-invariant since the following relations hold

$$[\boldsymbol{\alpha}^{ij}\Pi_j^{(int)}, S^{kl}] = i(\eta^{ik}\boldsymbol{\alpha}^{lj}\Pi_j^{(int)} - \eta^{il}\boldsymbol{\alpha}^{kj}\Pi_j^{(int)}),$$

$$[\varepsilon^{0d}{}_{a_1a_2a_3a_4\ldots a_{d-3}a_{d-2}}S^{a_1a_2}S^{a_3a_4}\ldots S^{a_{d-3}a_{d-2}}, S^{kl}] = 0,$$

for $i,j,k,l = 1,2,\ldots,d-1$. To prove that Eq.(61) holds on $\mathcal{W}$ it suffices to prove that Eq.(61) holds on any $SO(d-2)$-dominant weight state in $\mathcal{W}$. (By lemma V.1 this also implies that $\mathcal{W}$ is irreducible and the $SO(d-2)$-dominant weight state in $\mathcal{W}$ is therefore unique up to a scalar multiple.) We therefore choose any $SO(d-2)$-dominant weight state $|\Phi_l\rangle \in \mathcal{W}$. We designate its weight as $(l'_{n-1}, l'_{n-2}, \ldots, l'_2, l'_1)_r$. It follows from Eq.(60) that $\alpha = -\rho 2^{n-1}(n-1)!(l'_{n-1} + r(n-2))\ldots(l'_2+r)l'_1$.

From Eq.(26) it follows that

$$\begin{aligned}\boldsymbol{\alpha}^{1i'}|\Phi_l\rangle &= 2(n-2)\varepsilon^{0d}{}_{1i'2j'\ldots}S^{2j'}\ldots|\Phi_l\rangle = \\ &= 2r(n-2)\varepsilon^{0d}{}_{1i'2j'\ldots}iS^{1j'}\ldots|\Phi_l\rangle = -ir\boldsymbol{\alpha}^{2i'}|\Phi_l\rangle,\end{aligned} \tag{62}$$

for $i' = 3,5,\ldots,d-1$. If we add the equation with $i=2$ from Eq.(59), which we first multiply by $ir\boldsymbol{\alpha}^{12}$, to the equation with $i=1$, we find taking into account Eq.(62) that

$$0 = ((\Pi_2^{(int)} - ir\Pi_1^{(int)})\boldsymbol{\alpha}^{12} + \Pi_{i'}^{(int)}(\boldsymbol{\alpha}^{1i'} + ir\boldsymbol{\alpha}^{2i'}))|\Phi_l\rangle = (\Pi_2^{(int)} - ir\Pi_1^{(int)})\boldsymbol{\alpha}^{12}|\Phi_l\rangle. \tag{63}$$

Following similar techniques as in section III and taking into account Eq.(60), we find

$$\begin{aligned}\alpha|\Phi_l\rangle &= -\rho\varepsilon^{0d}{}_{a_1a_2a_3a_4\ldots}S^{a_1a_2}S^{a_3a_4}\ldots|\Phi_l\rangle = \\ &= -\rho 2(n-1)(S^{12} + r(n-2))\varepsilon^{0d}{}_{12a_1a_2\ldots}S^{a_1a_2}\ldots|\Phi_l\rangle = \\ &= -\rho 2(n-1)(l'_{n-1} + r(n-2))\boldsymbol{\alpha}^{12}|\Phi_l\rangle \\ \Longrightarrow \boldsymbol{\alpha}^{12}|\Phi_l\rangle &= \frac{\alpha}{-\rho 2(n-1)(l'_{n-1} + r(n-2))}|\Phi_l\rangle.\end{aligned} \tag{64}$$

Since the condition $l_1 \neq 0$ implies $\alpha \neq 0$ (Eq.(48)), we may conclude from Eq.(64) and Eq.(63)

$$(\Pi_2^{(int)} - ir\Pi_1^{(int)})|\Phi_l\rangle = 0. \tag{65}$$

From Eqs. (59), (62), (65) we find

$$
\begin{aligned}
0 &= (\Pi_1^{(int)}\boldsymbol{\alpha}^{1i'} + \Pi_2^{(int)}\boldsymbol{\alpha}^{2i'} + \Pi_{j'}^{(int)}\boldsymbol{\alpha}^{j'i'})|\Phi_l\rangle = \\
&= (-ri\Pi_1^{(int)}\boldsymbol{\alpha}^{2i'} + \Pi_2^{(int)}\boldsymbol{\alpha}^{2i'} + \Pi_{j'}^{(int)}\boldsymbol{\alpha}^{j'i'})|\Phi_l\rangle = \\
&= -ir[\Pi_1^{(int)}, \boldsymbol{\alpha}^{2i'}]|\Phi_l\rangle + \boldsymbol{\alpha}^{2i'}(-ir\Pi_1^{(int)} + \Pi_2^{(int)})|\Phi_l\rangle + \Pi_{j'}^{(int)}\boldsymbol{\alpha}^{j'i'}|\Phi_l\rangle = \\
&= -ir[\Pi_1^{(int)}, \boldsymbol{\alpha}^{2i'}]|\Phi_l\rangle + \Pi_{j'}^{(int)}\boldsymbol{\alpha}^{j'i'}|\Phi_l\rangle = \\
&= 2r(n-2)(\Pi_{j'}^{(int)}\varepsilon^{0d}{}_{12i'j'a_1a_2\ldots}S^{a_1a_2}\ldots)|\Phi_l\rangle + \Pi_{j'}^{(int)}\boldsymbol{\alpha}^{j'i'}|\Phi_l\rangle = \\
&= 2r(n-2)(\Pi_{j'}^{(int)}\varepsilon^{0d}{}_{12i'j'a_1a_2\ldots}S^{a_1a_2}\ldots)|\Phi_l\rangle + \\
&\quad +2r(n-2)(\Pi_{j'}^{(int)}(l'_{n-1} + r(n-3))\varepsilon^{0d}{}_{12i'j'a_1a_2\ldots}S^{a_1a_2}\ldots)|\Phi_l\rangle = \\
&= 2r(n-2)(l'_{n-1} + r(n-2))\Pi_{j'}^{(int)}(\varepsilon^{0d}{}_{12i'j'a_1a_2\ldots}S^{a_1a_2}\ldots)|\Phi_l\rangle
\end{aligned}
$$

with indices $i', j' = 3, 5, \ldots, d-1$.

We summarize

$$
\Pi_j^{(int)}(\varepsilon^{0d12ij}{}_{a_1a_2a_3a_4\ldots}S^{a_1a_2}S^{a_3a_4}\ldots)|\Phi_l\rangle = 0 \quad \text{for all } i. \tag{66}
$$

Now, we can repeat this process by using (27) to obtain

$$
\Pi_j^{(int)}(\varepsilon^{0d1234ij}{}_{a_1a_2a_3a_4\ldots}S^{a_1a_2}S^{a_3a_4}\ldots)|\Phi_l\rangle = 0 \quad \text{for all } i \tag{67}
$$

and so on. Finally we arrive at

$$
\Pi_j^{(int)}(\varepsilon^{0d1234\ldots(d-8)(d-7)ij}{}_{a_1a_2a_3a_4}S^{a_1a_2}S^{a_3a_4})|\Phi_l\rangle = 0 \quad \text{for all } i, \tag{68}
$$

$$
\Pi_j^{(int)}(\varepsilon^{0d1234\ldots(d-6)(d-5)ij}{}_{a_1a_2}S^{a_1a_2})|\Phi_l\rangle = 0 \quad \text{for all } i \tag{69}
$$

and

$$
\Pi_j^{(int)}(\varepsilon^{0d1234\ldots(d-4)(d-3)ij})|\Phi_l\rangle = 0 \quad \text{for all } i. \tag{70}
$$

This obviously implies

$$
\Pi_{d-1}^{(int)}|\Phi_l\rangle = 0, \quad \Pi_{d-2}^{(int)}|\Phi_l\rangle = 0. \tag{71}
$$

We use this result in Eq.(69) for $i = d-4, d-3$ to find

$$\Pi^{(int)}_{d-3} S^{d-2\;d-1}|\Phi_l\rangle = 0, \quad \Pi^{(int)}_{d-4} S^{d-2\;d-1}|\Phi_l\rangle = 0. \tag{72}$$

Since $S^{d-2\;d-1}|\Phi_l\rangle = l'_1|\Phi_l\rangle$ and $\alpha \neq 0$, Eq.(48) requires that $l'_1 \neq 0$ and we conclude from Eq.(72)

$$\Pi^{(int)}_{d-3}|\Phi_l\rangle = 0, \quad \Pi^{(int)}_{d-4}|\Phi_l\rangle = 0. \tag{73}$$

Using this result and the one of Eq.(71) in Eq.(68) for $i = d-6, d-5$, we find

$$\Pi^{(int)}_{d-5}\varepsilon^{0d1234\ldots(d-8)(d-7)(d-6)(d-5)}{}_{a_1a_2a_3a_4}S^{a_1a_2}S^{a_3a_4}|\Phi_l\rangle = 0,$$
$$\Pi^{(int)}_{d-6}\varepsilon^{0d1234\ldots(d-8)(d-7)(d-6)(d-5)}{}_{a_1a_2a_4a_4}S^{a_1a_2}S^{a_3a_4}|\Phi_l\rangle = 0.$$

Again, the equality $\varepsilon^{0d1234\ldots(d-8)(d-7)(d-6)(d-5)}{}_{a_1a_2a_3a_4}S^{a_1a_2}S^{a_3a_4}|\Phi_l\rangle = 8(l'_2+r)l'_1|\Phi_l\rangle$ and $l'_1 \neq 0$ implies that

$$\Pi^{(int)}_{d-5}|\Phi_l\rangle = 0, \quad \Pi^{(int)}_{d-6}|\Phi_l\rangle = 0 \tag{74}$$

and so on. Finally we find

$$\Pi^{(int)}_{1}|\Phi_l\rangle = 0, \quad \Pi^{(int)}_{2}|\Phi_l\rangle = 0, \tag{75}$$

which proves $\Pi^{(int)}_i|\Phi_l\rangle = 0$ for $i = 1, 2, \ldots, d-1$ and concludes the proof. ■

We may now write down the main result of this paper:

On the space with internal Lorentz group $SO(1, d-1)$ representation $(l_n, l_{n-1}, \ldots, l_1)_{+1}$, $(l_1 \neq 0)$ the equation

$$(W^a = \alpha p^a)|\Phi\rangle, \quad \text{with} \quad \alpha = -\rho 2^{n-1}(n-1)!(l_{n-1}+n-2)\ldots(l_2+1)l_1 \tag{76}$$

is the equation of motion for massless particles corresponding to the following representations of the Poincaré group

$$(l_{n-1}, \ldots, l_1; +1)_{+1} \quad \text{and} \quad (l_{n-1}, \ldots, -l_1; -1)_{+1}, \tag{77}$$

where the masslessness condition $(p_ap^a - 0)|\Phi\rangle$ is not needed, since it follows from (76).

With the aid of Eqs. (36), (37) the equation of motion can also be written as

$$(W^a = |\alpha|\Gamma^{(int)}p^a)|\Phi\rangle, \quad |\alpha| = \rho 2^{n-1}(n-1)!(l_{n-1}+n-2)\dots(l_2+1)|l_1|. \tag{78}$$

This equation is convenient when dealing with positive and negative handedness on the same footing (an example of this is the Dirac equation) since $|\alpha|$ is independent of the sign of l_1.

We note that the particular value of ρ is irrelevant in Eqs. (76), (78) since ρ is found in both the lefthand and the righthand side of the equations and thus cancels out. The value of ρ becomes relevant when dealing with the particular spin (i.e. in the next section) when it is used to insure that the operators S^i have the familiar values independent of the dimension.

Making a choice of $a = 0$ one finds

$$(\overrightarrow{S}.\overrightarrow{p} = |\alpha|\Gamma^{(int)}p^0)|\Phi\rangle. \tag{79}$$

Since $(\Gamma^{(int)})^2 = 1$, one immediately finds that $|\alpha| = |\overrightarrow{S}.\overrightarrow{p}|/|p^0|$. We shall comment that for spinors Eq.(79) is equivalent to Eq.(78), while for vectors Eq.(78) gives additional condition to Eq.(79).

VI. DISCUSSIONS ON EXAMPLES.

We have shown in previous sections that massless fields in $d = 2n$-dimensional spaces, if having the Poincaré symmetry, obey the linear equations of motion. We shall present in this section the equations of motion for massless fields in $d = 2n$ dimensional spaces for two types of fields, which we call [3–8] spinorial and vectorial fields, respectively. Spinorial fields are defined by the generators of the Lorentz transformations, which fulfil Eq.(7) and are represented by $(\frac{1}{2},\dots,\pm\frac{1}{2})_{+1}$. The irreducible representations of the internal Lorentz group for vectorial fields are given by $(1,\dots,\pm 1)_{+1}$. We shall later define the generators of the Lorentz group for the vectorial internal space in any dimension $d = 2n$ (see Eq.(85)).

Although both kinds of fields can be treated in an ordinary way, that is by using the group theoretical approaches, which determine properties of states of an irreducible representation by defining the operation of the generators of the group on a state without going

into any representation, just as we have done in sections above, we shall use the space of anicommuting $d = 2n$ coordinates to describe the internal degrees of freedom (it is the spin degree of freedom in our case) for both types of fields mentioned above, representing states as polynomials of the anticommuting coordinates. We shall do that, because this way offers a very simple and transparent presentation of operators in any dimension, as well as of representations. We shall follow the approach of one of us [3,4,7], using Grassmann coordinates. As it was proven in ref. [9] we could as well use differential forms instead of Grassmann coordinates. At the end we shall comment on results.

We briefly review the description of the internal space we use [7]. It is the Grassmann space (also known as the exterior algebra) spanned over $2n$-dimensional vector space; we denote it as Λ_{2n}. Formally, this space is the space spanned by $2n$ variables

$$\theta^0, \theta^1, \dots, \theta^{2n} \tag{80}$$

and their products, where the following anticommutation relations hold

$$\{\theta^a, \theta^b\} = \theta^a\theta^b + \theta^a\theta^b = 0, \quad a, b = 0, 1, \dots, 2n. \tag{81}$$

We use the symbol θ^a to also denote the operator of multiplication with the variable θ^a. Accordingly, relations (81) hold for θ^a as operators also. The operator of differentiation $\frac{\partial}{\partial\theta^a}$ is defined as follows

$$\frac{\partial}{\partial\theta^a}(\theta^{a_1}\dots\theta^{a_p}) = \begin{cases} 0, & \text{if } a \neq a_1, \dots, a_k, \\ (-1)^{l-1}\theta^{a_1}\dots\theta^{a_{l-1}}\theta^{a_{l+1}}\dots\theta^{a_p}, & \text{if } a = a_l \end{cases}, \tag{82}$$

if we assume that the differentiation is allways performed from the left.

We denote $p_{\theta a} = -i\frac{\partial}{\partial\theta^a}$. It is easily checked that the following anticommutation relations hold

$$\{\theta^a, \theta^b\} = \{p^a_\theta, p^b_\theta\} = 0, \quad \{\theta^a, p^b_\theta\} = -i\eta^{ab}. \tag{83}$$

Following the references [3,7], we define the generators of the internal Lorentz group. For spinors they are

$$S_s^{ab} = -\frac{i}{4}[a^a, a^b], \quad \text{where} \quad a^a = i(p_\theta^a - i\theta^a), \tag{84}$$

with $[a^a, a^b] = a^a a^b - a^b a^a$, and for vectors they are

$$S_v^{ab} = \theta^a p_\theta^b - \theta^b p_\theta^a. \tag{85}$$

Both S_s^{ab} and S_v^{ab} satisfy Lorentz group commutation relations and therefore furnish a representation of the Lorentz group. For operators a^a the following anticommutation relations hold

$$\{a^a, a^b\} = 2\eta^{ab}. \tag{86}$$

We shall first comment on the representations for the spinorial case and derive the corresponding equations of motion for spinorial massless fields in $d = 2n$. Later we shall do the same for vectorial massless fields.

A. Spinors.

We introduce the following definitions for the handedness operator and the Pauli-Ljubanski vector ($\beta = i\frac{2^n}{d!}, \rho = \frac{2^{n-2}}{(d-2)!}$)

$$\Gamma^{(int)} = i\frac{2^n}{d!}\varepsilon_{a_1 a_2 \ldots a_{d-1} a_d} S_s^{a_1 a_2} S_s^{a_3 a_4} \ldots = -(-i)^{n+1} a^0 a^1 \ldots a^{2n}, \tag{87}$$

$$W^a = \frac{2^{n-2}}{(d-2)!}\varepsilon^{a a_1}{}_{a_2 a_3 \ldots a_{d-1}} p_{a_1} S_s^{a_2 a_3} S_s^{a_4 a_5} \ldots = \frac{(-i)^{n-1}}{2} p_b a^a a^b \ (a^0 a^1 \ldots a^{2n}). \tag{88}$$

This ensures ${\Gamma^{(int)}}^2 = 1$ which implies $l_1 \neq 0$ for all irreducible representations $(l_n, \ldots, l_1)_{+1}$ into which the representation space decomposes. Then we must have $|l_1| \geq \frac{1}{2}$ and by the equation (36) this implies $|\Gamma^{(int)}| \geq 1$ where equality is achieved for $l_n = l_{n-1} = \ldots = |l_1| = \frac{1}{2}$. Since this is our case we conclude that we are dealing with the representations $(\frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})_{+1}$. For $d = 6$ we present in Table I the basis for two of the irreducible representations $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$ our internal space decomposes into.

We now proceed to determination of the equations of motion. First we introduce the operators

$$\bar{S}_s{}^{ab} = \frac{2^{n-2}}{(d-2)!}\varepsilon^{ab}{}_{a_1 a_2 \ldots a_{d-2}} S_s^{a_1 a_2} S_s^{a_3 a_4} \ldots = \frac{(-i)^{n-1}}{2} a^a a^b \ (a^0 a^1 \ldots a^{2n}). \tag{89}$$

and state

Lemma VI A.1 : It holds

$$\bar{S}_s{}^{ab} = -i\Gamma^{(int)} S_s^{ab}. \tag{90}$$

Proof: Obvious from our previous definitions. ■

We now state our equation of motion. It is Eq.(78) and it reads

$$(W^a = p_b \bar{S}_s{}^{ab} = \frac{1}{2}\Gamma^{(int)} p^a)|\Phi\rangle. \tag{91}$$

Now, the lemma VI A.1 implies that this is equivalent to

$$(-ip_b \Gamma^{(int)} S_s^{ab} = \frac{1}{2}\Gamma^{(int)} p^a)|\Phi\rangle. \tag{92}$$

Taking into account the definition $S_s^{ab} = -\frac{i}{2}(a^a a^b - \eta^{ab})$ and multiplying the above equation by $2\Gamma^{(int)}$ we obtain

$$(-p_b(a^a a^b - \eta^{ab}) = p^a)|\Phi\rangle \tag{93}$$

which upon multiplication by a_a implies

$$p_b a^b |\Phi\rangle = 0. \tag{94}$$

By reversing this process we may show that Eq. (94) implies Eq. (78). It is therefore our equation of motion. For $d = 4$ we may recognize it as the Weyl-like equation. Further discussions about this equation of motion can be found in references [7,9] and other references included in these papers. We may also check, that by multiplying Eq.(94) by $-(-i)^{n+1}\prod_{a\neq 0} a^a$, Eq.(94) is equivalent to

$$(\Gamma^{(int)} p^0 = 2\overrightarrow{S_s}.\overrightarrow{p})|\Phi\rangle. \tag{95}$$

Defining γ^a matrices for any [17,9] d and replacing a^a by γ^a, we could repeat all the above derivations and end up with the Dirac equation for spinors in d-dimensional spaces.

B. Vectors.

We take the following definition of the handedness operator ($\beta = \frac{i}{2^n n!^2}$)

$$\Gamma^{(int)} = \frac{i}{2^n n!^2}\varepsilon_{a_1 a_2 \ldots a_{d-1} a_d} S_v^{a_1 a_2} S_v^{a_3 a_4} \ldots = \frac{i}{n!^2}\varepsilon_{a_1 a_2 \ldots a_{d-1} a_d}\theta^{a_1} p_\theta^{a_2}\theta^{a_3} p_\theta^{a_4} \ldots . \tag{96}$$

We limit our attention to the subspace Λ_n spanned by n-monomials of the form

$$\theta^{a_1}\theta^{a_2}\ldots\theta^{a_n}, \quad a_1, \ldots, a_n = 0, 1, \ldots, 2n. \tag{97}$$

On this subspace it obviously holds

$$((\Gamma^{(int)})^2 = 1)|\Phi\rangle, \tag{98}$$

from where we conclude that $l_1 \neq 0$ for all irreducible representations $(l_n, \ldots, l_1)_{+1}$ into which Λ_n decomposes. It is easily checked that

$$(S_v^{ab} S_{v\,ab})^2 = S_v^{ab} S_{v\,ab}, \quad \text{(no summation over } a, b) \tag{99}$$

which implies that (in absolute value) the possible values for S_v^{ab} are $0, 1$. Since l_1 is nonzero, we conclude from Eq.(36) that $|\Gamma^{(int)}| \geq 1$ where equality is achieved for

$$l_n = l_{n-1} = \ldots = |l_1| = 1. \tag{100}$$

Therefore, equation (98) implies Eq.(100). For $d = 6$ we present in Table II the basis for the irreducible representation $(1, 1, 1)$.

We now proceed to the formulation of the equation of motion on Λ_n. We take $\rho = \frac{1}{2^{n-1}(n-1)!^2}$ in the definition of the Pauli-Ljubanski vector

$$\begin{aligned} W^a &= \frac{1}{2^{n-1}(n-1)!^2}\varepsilon^{aa_1}{}_{a_2 a_3 \ldots a_{d-1}} p_{a_1} S_v^{a_2 a_3} S_v^{a_4 a_5} \ldots = \\ &= \frac{1}{(n-1)!^2}\varepsilon^{aa_1}{}_{a_2 a_3 \ldots a_{d-1}} p_{a_1}\theta^{a_2} p_\theta^{a_3}\theta^{a_4} p_\theta^{a_5} \ldots = p_b \bar{S_v}^{ab} \end{aligned} \tag{101}$$

where we have introduced

$$\bar{S_v}^{ab} = \frac{1}{(n-1)!^2}\varepsilon^{ab}{}_{a_1 a_2 \ldots a_{d-3} a_{d-2}}\theta^{a_1} p_\theta^{a_2}\theta^{a_3} p_\theta^{a_4} \ldots . \tag{102}$$

Before formulating the equations of motion we prove

Lemma VI B.1 : On the space Λ_n it holds

$$(\bar{S}_v{}^{ab} = -i\Gamma^{(int)} S_v^{ab})|\Phi\rangle. \tag{103}$$

Proof: We fix $a, b = 0, 1, \ldots, 2n$ and write

$$\Gamma^{(int)} = \frac{i}{n!^2}\Big(n(n-1)p_\theta^a p_\theta^b \varepsilon_{aba_1a_2\ldots a_{d-2}}\theta^{a_1}\theta^{a_2}\theta^{a_3}p_\theta^{a_4}\ldots + \\ + n(n-1)\theta^a\theta^b\varepsilon_{aba_1a_2\ldots a_{d-2}}p_\theta^{a_1}p_\theta^{a_2}\theta^{a_3}p_\theta^{a_4}\ldots + n^2 S_v^{ab}[\bar{S}_{v\,ab}(n-1)!^2]\Big). \tag{104}$$

Multiplying this by $S_{v\,ab} = \theta_a p_{\theta b} - \theta_b p_{\theta a}$ and $\bar{S_v}ab$ respectively we obtain

$$\Gamma^{(int)} S_{v\,ab} = i(S_v^{ab} S_{v\,ab})\bar{S}_{v\,ab}, \quad \text{(no summation over } a, b) \tag{105}$$

$$\Gamma^{(int)} \bar{S}_{v\,ab} = i(\bar{S}_v{}^{ab}\bar{S}_{v\,ab})S_{v\,ab}. \quad \text{(no summation over } a, b) \tag{106}$$

Taking into account the equations (98), (99), (105), (106) we obtain

$$\begin{aligned} i\bar{S}_{v\,ab}|\Phi\rangle &= (i\bar{S}_{v\,ab}(1 - S_v^{ab}S_{v\,ab}) + i\bar{S}_{v\,ab}S_v^{ab}S_{v\,ab})|\Phi\rangle = \\ &= (-\Gamma^{(int)}\bar{S}_{v\,ab}\bar{S}_v{}^{ab}S_{v\,ab}(1 - S_v^{ab}S_{v\,ab}) + \Gamma^{(int)}S_{v\,ab})|\Phi\rangle = \\ &= (-i\bar{S}_{v\,ab}\bar{S}_v{}^{ab}\bar{S}_{v\,ab}(S_v^{ab}S_{v\,ab})(1 - S_v^{ab}S_{v\,ab}) + \Gamma^{(int)}S_{v\,ab})|\Phi\rangle = \\ &= \Gamma^{(int)}S_{v\,ab}|\Phi\rangle, \quad \text{(no summation over } a, b). \end{aligned}$$

■

We may now write our equations of motion (78)

$$(p_b\bar{S}_v{}^{ab} = \Gamma^{(int)}p^a)|\Phi\rangle. \tag{107}$$

With the equation (103) this is equivalent to

$$(p_b S_v^{ab} = ip^a)|\Phi\rangle \tag{108}$$

or

$$(p_b(\theta^a p_\theta^b - \theta^b p_\theta^a) = ip^a)|\Phi\rangle. \tag{109}$$

We multiply the last equation by $p_\theta^a \theta_a$ (no summation over a) to obtain

$$(p_\theta^a (p_b \theta^b) = 0)|\Phi\rangle. \tag{110}$$

Since this holds for every a we may conclude

$$(p_b \theta^b = 0)|\Phi\rangle. \tag{111}$$

We similarly obtain

$$(p_b p_\theta^b = 0)|\Phi\rangle. \tag{112}$$

The equations (111), (112) are also easily shown to imply (108) : one simply multiplies Eqs. (111), (112) by $p_{\theta a}, \theta_a$, respectively, and adds the two equations. Therefore (111) and (112) are the equations of motion.

The above discussion also shows that Eq.(79), which is a special case $a = 0$ of Eq.(78), is not equivalent to Eq.(78) for vectors. It is equivalent to (110) with $b = 0$ which in general does not imply (111). Therefore in the case of vectors (and in general) Eq.(79) must be complemented by additional equations to ensure that it is the equation of motion. These additional equations are of course the remaining equations $a = 1, 2, \ldots$ of (78).

Finally we show that the obtained equations of motion are the generalized Maxwell equations [15]. The space Λ_n may be identified with the space of totally antisymmetric n-tensor $F^{a_1 a_2 \ldots a_n}$ with the correspondence [18]

$$\theta_{a_1}\theta_{a_2}\ldots\theta_{a_n} \in \Lambda_n \longleftrightarrow F^{c_1 c_2 \ldots c_n} = \frac{1}{n!}\varepsilon^{c_1 c_2 \ldots c_n}{}_{b_1 b_2 \ldots b_n}\varepsilon^{b_1 b_2 \ldots b_n}{}_{a_1 a_2 \ldots a_n}. \tag{113}$$

Then Eqs.(111), (112) may be written as

$$p_a F^{a a_1 \ldots a_{n-1}} = p_a \varepsilon^{a a_1 \ldots a_{n-1}}{}_{b_1 b_2 \ldots b_n} F^{b_1 b_2 \ldots b_n} = 0, \tag{114}$$

which are the generalized Maxwell equations.

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TABLES

TABLE I. Two of the spinorial irreducible representations of the Lorentz group with $(\frac{1}{2}, \frac{1}{2}, \frac{\pm 1}{2})$ in the Grassmann space for $d = 6$. The dominant weight state is listed first, so the first representations corresponds to $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the second to $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

$\vert\Phi\rangle$	S_s^{06}	S_s^{12}	S_s^{35}	$\Gamma^{(int)}$
$(\theta^0 + \theta^6)(\theta^1 - i\theta^2)(\theta^3 - i\theta^5)$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1
$(1 + \theta^0\theta^6)(\theta^1 - i\theta^2)(1 + i\theta^3\theta^5)$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1
$(\theta^0 + \theta^6)(1 + i\theta^1\theta^2)(1 + i\theta^3\theta^5)$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$(1 + \theta^0\theta^6)(1 + i\theta^1\theta^2)(\theta^3 - i\theta^5)$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1
$(\theta^0 + \theta^6)(\theta^1 - i\theta^2)(\theta^3 + i\theta^5)$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1
$(1 + \theta^0\theta^6)(\theta^1 - i\theta^2)(1 - i\theta^3\theta^5)$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$(\theta^0 + \theta^6)(1 + i\theta^1\theta^2)(1 - i\theta^3\theta^5)$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1
$(1 + \theta^0\theta^6)(1 + i\theta^1\theta^2)(\theta^3 + i\theta^5)$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1

TABLE II. One of the vectorial irreducible representations with $(1,1,1)$ of the Lorentz group in the Grassmann space for $d=6$. The dominant weight state is listed first.

$\vert\Phi\rangle$	S_v^{06}	S_v^{12}	S_v^{35}	$\Gamma^{(int)}$
$(\theta^0+\theta^6)(\theta^1-i\theta^2)(\theta^3-i\theta^5)$	i	1	1	-1
$(\theta^0+\theta^6)(\theta^1+i\theta^2)(\theta^3+i\theta^5)$	i	-1	-1	-1
$(\theta^0-\theta^6)(\theta^1-i\theta^2)(\theta^3+i\theta^5)$	$-i$	1	-1	-1
$(\theta^0-\theta^6)(\theta^1+i\theta^2)(\theta^3-i\theta^5)$	$-i$	-1	1	-1
$(\theta^0\theta^6+i\theta^1\theta^2)(\theta^3-i\theta^5)$	0	0	1	-1
$(\theta^0\theta^6+i\theta^3\theta^5)(\theta^1-i\theta^2)$	0	1	0	-1
$(\theta^0+\theta^6)(\theta^1\theta^2-\theta^3\theta^5)$	i	0	0	-1
$(\theta^0\theta^6-i\theta^1\theta^2)(\theta^3+i\theta^5)$	0	0	1	-1
$(\theta^0\theta^6-i\theta^3\theta^5)(\theta^1+i\theta^2)$	0	1	0	-1
$(\theta^0-\theta^6)(\theta^1\theta^2+\theta^3\theta^5)$	i	0	0	-1